

Phase-Matching Condition for Superpositions of Coherent States in a Mach-Zehnder Interferometer

Xiang Zhao,¹ Luyi Shen,¹ Xiaoxing Jing,¹ Jing Liu,¹ and Xiaoguang Wang^{1,2,*}

¹*Zhejiang Institute of Modern Physics, Department of Physics, Zhejiang University, Hangzhou 310027, China*

²*Synergetic Innovation Center of Quantum Information and Quantum Physics, University of Science and Technology of China, Hefei, Anhui 230026, China*

We discussed the phase-matching condition of the input states for a specific Mach-Zehnder interferometer to enhance the phase sensitivity. The input states are a coherent state and a superposition of coherent states. For both cases with and without photon losses, the phase-matching condition to enhance phase sensitivity is found to be unchanged no matter the input state has parity or not.

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I. INTRODUCTION

Phase estimation is an important task in quantum metrology. Classically the measurement precision in optical phase estimation is bounded by the standard quantum limit (SQL), $1/\sqrt{N}$, where N is the total number of photons. Such sensitivity, however, can be further enhanced by exploiting spin-squeezing techniques [1]. In 1981, Caves [2] pointed out that, by feeding a high intensity coherent state in one input port of a interferometer and a low intensity squeezed vacuum state in the other, the phase sensitivity can approach the Heisenberg scaling, $1/N$. The SQL can also be beat by employing non-classical entangled states [3, 4]. The realization of surpassing the SQL as well as approaching the Heisenberg limit have brought about significant progress in recent years [5–7].

In Ref. [2], Caves also pointed out that, in order to enhance precision of measurement, the phase difference of the input states should satisfy a certain relation. Such relation is called the phase-matching condition (PMC). Recently a more general PMC for even (odd) states has been investigated [8]. However, it remains unknown whether non-even (odd) states also have a general PMC invariance. Therefore, the major purpose of this paper is to study the PMC of non-even (odd) states, especially under particle losses.

In this paper, we discuss a specific Mach-Zehnder interferometer. One of the input port is injected by a coherent state and the other is injected by a coherent superposition state whose parity depends on an adjustable parameter. We first give an analytic expression of the QFI without particle losses and determine a PMC to optimize the parameter precision. Then we consider a setting where particle losses occur in both arms with the same transmission coefficient and give the expression of the QFI as well as the PMC. Based on the expressions of the QFIs, we prove that the PMC keeps unchanged for any transmission coefficients, whether input state is with parity or

not.

II. MACH-ZEHNDER INTERFEROMETER

Mach-Zehnder (MZ) interferometer is a well-known optical apparatus in quantum metrology. It can be used to study many counterintuitive phenomena in quantum mechanics [9, 10]. A general MZ interferometer mainly consists of beam splitters and phase shifts. A SU(2) MZ interferometer can be described by Schwinger operators, which are [11–13]

$$\begin{aligned} J_x &= \frac{1}{2}(a^\dagger b + b^\dagger a), \\ J_y &= \frac{1}{2i}(a^\dagger b - b^\dagger a), \\ J_z &= \frac{1}{2}(a^\dagger a - b^\dagger b), \end{aligned} \quad (1)$$

where a , a^\dagger , b , b^\dagger are annihilation and creation operators for ports A and B. Above Schwinger operators satisfy the commutation relations

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad (2)$$

where ϵ_{ijk} is the Levi-Civita symbol.

A 50:50 beam splitter can be described by a unitary transformation [11–13]

$$B_x = \exp\left(-i\frac{\pi}{2}J_x\right), \quad (3)$$

and the phase shift is

$$P_z = \exp(i\theta J_z). \quad (4)$$

where θ is the phase difference between two arms after phase shift. With Eqs. (3) and (4), the total transformation performed by the MZ interferometer can be expressed by

$$U_{\text{MZ}} = B_x P_z B_x^\dagger = \exp(-i\theta J_y). \quad (5)$$

A vivid picture for the performance of the SU(2) MZ interferometer can be given [11] from the angular momentum description of the Schwinger operators. The operators B_x , B_x^\dagger and P_z can also be treated as the rotation

*Electronic address: xgwang@zimp.zju.edu.cn

operation to the input states about the x axis and the z axis, respectively. Combining the three operations in sequence, the transformation of total setup is equivalent to a rotation about the y axis.

III. QUANTUM FISHER INFORMATION

Fisher information, initially introduced by Fisher [14] in 1925, is a very important quantity in statistics. Quantum Fisher information (QFI) is the quantum extension of Fisher information and of great significance in quantum technology and quantum metrology [7, 15–17]. In quantum metrology, it is a central concept to describe the lowest bound for the precision of a parameter under estimation. The variance of an unbiased estimator $\hat{\theta}$ (i.e., $\langle \hat{\theta} \rangle = \theta$) for the parameter θ is theoretically bounded by the quantum Cramér-Rao inequality: $\text{Var}(\theta) \geq 1/\nu F$ [18–21], where $\text{Var}(\cdot)$ is the variance, ν is the repeated times of experiments and F refers to quantum Fisher information. The quantum Fisher information is defined by $F = \text{Tr}(\rho L_\theta^2)$ [18, 19], where L_θ is so-called symmetric logarithmic derivative (SLD) and is determined by the equation $\partial_\theta \rho_\theta = (\rho_\theta L_\theta + L_\theta \rho_\theta)/2$.

Recently, it has been found that the quantum Fisher information for a non-full rank density matrix is determined by its support [22–24]. Denote the spectral decomposition $\rho = \sum_{i=1}^M p_i |\psi_i\rangle\langle\psi_i|$, where M is the dimension of the support, and p_i , $|\psi_i\rangle$ are the i th eigenvalue and eigenstate for ρ , respectively. In this representation, the quantum Fisher information can be expressed by [22–24]

$$F = \sum_{i=1}^M 4p_i \langle \partial_\theta \psi_i | \partial_\theta \psi_i \rangle - \sum_{i,j=1}^M \frac{8p_i p_j}{p_i + p_j} |\langle \partial_\theta \psi_i | \psi_j \rangle|^2. \quad (6)$$

IV. PMC WITHOUT PHOTON LOSSES

First we consider an ideal Mach-Zehnder interferometer without particle losses. For a pure input state $\rho_{\text{in}} = |\phi\rangle\langle\phi|$, the output state is $\rho_{\text{out}} = U_{\text{MZ}}|\phi\rangle\langle\phi|U_{\text{MZ}}^\dagger$. Based on Eq. (5), one can easily find that the QFI is

$$F = 4 (\langle \phi | J_y^2 | \phi \rangle - |\langle \phi | J_y | \phi \rangle|^2). \quad (7)$$

Utilizing the expression of J_y , the QFI can be expressed by

$$F = 2\bar{n}_A \bar{n}_B + \bar{n}_A + \bar{n}_B - 2\text{Re}(\langle a^\dagger b^2 \rangle) - 4|\langle J_y \rangle|^2, \quad (8)$$

where $\bar{n}_A = \text{Tr}(\rho_A a^\dagger a)$ and $\bar{n}_B = \text{Tr}(\rho_B b^\dagger b)$ are average photon numbers for arm A and B, respectively. The expected value $\langle \cdot \rangle := \langle \phi | \cdot | \phi \rangle$.

In this paper, we choose the input state in port A to be a coherent state $|i\alpha e^{i\phi}\rangle\langle i\alpha e^{i\phi}|$ and that in port B to be a coherent superposition state $|\alpha\rangle_+ |\alpha\rangle$, where $|\alpha\rangle_+ = N_\alpha(|\alpha\rangle + e^{i\omega}|\alpha\rangle)$ with $|\alpha\rangle$ also a coherent state and $N_\alpha^2 = 1/(2 + 2e^{-2|\alpha|^2} \cos \omega)$ the normalized number, $\omega \in$

$[0, \pi)$. Here the relative phase between two input ports is $\phi + \pi/2$, and $\phi + \pi/2 \in [0, \pi)$, so $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Notice that when $\omega = 0$, ρ_B owns parity, i.e., it is a even state satisfying $+\langle \alpha | b | \alpha \rangle_+ = 0$. After obtaining the terms,

$$\bar{n}_A = |\alpha|^2, \quad \bar{n}_B = |\alpha|^2 \frac{1 - e^{-2|\alpha|^2} \cos \omega}{1 + e^{-2|\alpha|^2} \cos \omega}, \quad (9)$$

$$\langle a^{\dagger 2} \rangle = -(\alpha^*)^2 e^{-2i\phi}, \quad \langle b^2 \rangle = \alpha^2, \quad (10)$$

$$\langle \phi | J_y | \phi \rangle = -2|\alpha|^2 N_\alpha^2 e^{-2|\alpha|^2} \sin \phi \sin \omega, \quad (11)$$

the QFI becomes

$$F = 2\bar{n}_A \bar{n}_B + \bar{n}_A + \bar{n}_B + 2|\alpha|^4 \cos(2\phi) - 16|\alpha|^4 N_\alpha^4 e^{-4|\alpha|^2} \sin^2 \omega \sin^2 \phi, \quad (12)$$

From this equation, it is easy to see that with fixed α , the PMC to maximum the QFI will always be satisfied as $\phi = 0$, regardless of the value of ω , that is, no matter input state has parity or not, the PMC does not change.

When $\phi = 0$, the maximal QFI F_m becomes

$$F_m = 2\bar{n}_A \bar{n}_B + \bar{n}_A + \bar{n}_B + 2\bar{n}_A^2. \quad (13)$$

Note that under the PMC, F_m only depends on the average photon numbers of both ports: \bar{n}_A and \bar{n}_B . And to investigate the relationship between the QFI and the phase sensitivity limit, we can also express F_m in terms of total photon number N by denoting $N = \bar{n}_A + \bar{n}_B$. With Eq. (9), N can be expressed as

$$N = \frac{2\bar{n}_A}{1 + e^{-2|\alpha|^2} \cos \omega}. \quad (14)$$

Then F_m becomes

$$F_m = N + 2\bar{n}_A N = N + (1 + e^{-2|\alpha|^2} \cos \omega) N^2. \quad (15)$$

Large F is required for the enhancement of phase sensitivity, and this needs high intensity of input states, which means $|\alpha| \gg 1$. Then with high intensity and suppose the total photon number is fixed, it is not difficult to find that $F_m \propto N + N^2$. Under this condition, F_m can lead to approach the Heisenberg scaling [8]. More interestingly, when $|\alpha|$ is large enough, the term which contains $\cos \omega$ will become neglectable, indicating that the parity of input states not only does not affect the PMC, but also makes no difference to the value of F_m under high intensity.

V. PMC WITH PHOTON LOSSES

In section IV we have discussed the PMC without particle losses, which is an ideal scenario. Now we will take into account the effects of photon losses on PMC. Essentially, one can use a fictitious beam splitter to describe the photon losses process. In this paper we will

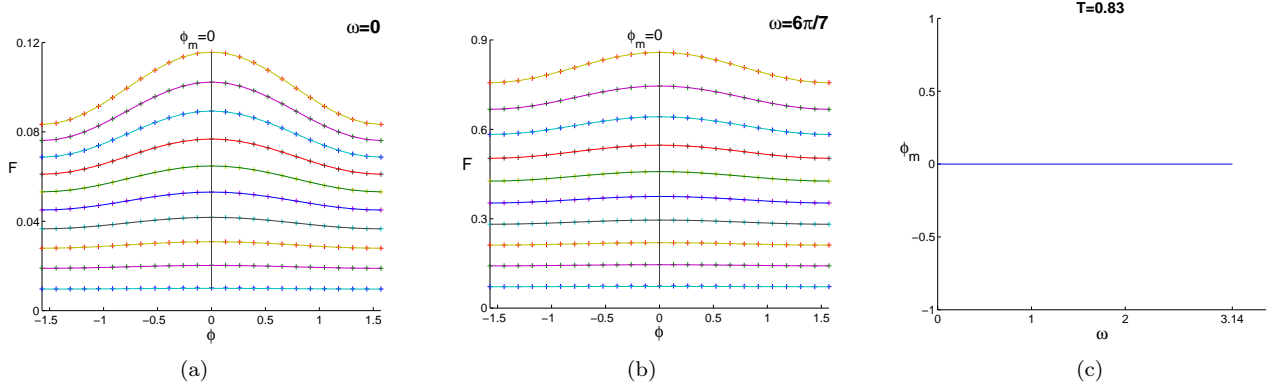


Figure 1: (Color online) Numeric and analytic results of QFI under particle losses. In figure (a) and (b) ω is fixed, and is chosen to be 0 and $6\pi/7$. The '+' dots represent the numerical results while the lines represent the analytic ones. Each plot has ten lines, which separately has, from bottom to top, the transmission coefficient T varying from 0.1 to 1. Figure (c) shows the ϕ_m for different ω with T being fixed at 0.83. The straight line demonstrates that ϕ_m is invariant against ω , namely, the PMC is unaffected by the input state's parity. α is set to be 0.3 in all three graphs.

use $B^T = \exp[i(2 \arccos \sqrt{T})J_x]$ to describe both real and fictitious beam splitters, where T is the so-called transmission coefficient. As for fictitious beam splitters, when $T = 1$, there are no particle losses, which goes back to our previous discussion; when $T = 0$, all the photons leak out of the interferometer. For convenience we also define the reflection coefficient R , which satisfies $T + R = 1$. Note that the operator B^T actually has no difference from the general form of Eq. (3), that is

$$B_x^T = \exp(i\tau J_x). \quad (16)$$

We just substitute τ with $\arccos \sqrt{T}$, i.e., $T = \cos^2 \frac{\tau}{2}$. Essentially, the whole photon loss setting can be converted into a neat scenario, which includes three steps: (1) the input state imports into a MZ interferometer without any loss on photons and passes the first beam splitter; (2) the output state goes through a particle loss channel; (3) the final state after particle loss undergoes a phase shift and passes the second beam splitter.

For simplicity, we assume that the transmission coefficient T are the same on both arms and the input state is separable: $\rho_{\text{in}} = \rho_A \otimes \rho_B$. Then after the first 50:50 beam splitter $B^{\frac{1}{2}}$ we have

$$\rho_0 = B^{\frac{1}{2}} \rho_{\text{in}} B^{\frac{1}{2}\dagger}, \quad (17)$$

where $B^{\frac{1}{2}} = \exp[i(2 \arccos \sqrt{\frac{1}{2}})J_x]$. Then the state goes through a particle loss process, after which we have

$$\rho = \text{Tr}_{CD} \left(B_{AC}^T B_{BD}^T \rho_0 B_{BD}^{T\dagger} B_{AC}^{T\dagger} \right), \quad (18)$$

where $B_{AC}^T = \exp[i(2 \arccos \sqrt{T})J_x^{AC}]$ and $B_{BD}^T = \exp[i(2 \arccos \sqrt{T})J_x^{BD}]$. Here $J_x^{AC} = \frac{1}{2}(a^\dagger c + a c^\dagger)$ and $J_x^{BD} = \frac{1}{2}(b^\dagger d + b d^\dagger)$ with $c, c^\dagger, d, d^\dagger$ the annihilation and creation operators of mode C and D. Trace operation is conducted here to eliminate the information on mode C

and mode D. And consequently we need to cope with mixed states afterwards. Since the second beam splitter merely causes a rotation of ρ in the Hilbert space, it does not affect the QFI, and we shall omit it in the following discussion.

Now we continue to use the previous states as input states, that is, $\rho_A = |i\alpha e^{i\phi}\rangle\langle i\alpha e^{i\phi}|$ and $\rho_B = |\alpha\rangle_+ \langle \alpha|$, where $|\alpha\rangle_+ = N_\alpha(|\alpha\rangle + e^{i\omega}|\alpha\rangle)$ and $N_\alpha^2 = 1/(2 + 2e^{-2|\alpha|^2} \cos \omega)$, $\omega \in [0, \pi)$. After some algebra, we can express ρ in the following form (see Appendix A)

$$\rho = \begin{pmatrix} \eta & \xi e^{i\tau} \\ \xi e^{-i\tau} & 1 - \eta \end{pmatrix}, \quad (19)$$

where $\eta = N_\alpha^2(1 + 2e^{-2|\alpha|^2} \cos \omega + p_t^2)$, $\xi e^{i\tau} = N_\alpha^2(p_r e^{-i\omega} + p_t) \sqrt{1 - p_t^2}$. Here $p_t = \exp(-2|\alpha|^2 T)$, $p_r = \exp(-2|\alpha|^2 R)$. In addition, notice that since the second beam splitter is omitted from the calculation, now we should use the operator J_z instead of J_y , and the expression of the QFI becomes

$$F = \sum_{i=\pm} 4\lambda_i \langle \lambda_i | J_z^2 | \lambda_i \rangle - \sum_{i,j=\pm} \frac{8\lambda_i \lambda_j}{\lambda_i + \lambda_j} |\langle \lambda_i | J_z | \lambda_j \rangle|^2, \quad (20)$$

where λ_i and $|\lambda_i\rangle$ are eigenvalues and eigenstates of ρ . It is easy to find that [25]

$$\lambda_{\pm} = \frac{1 \pm \sqrt{1 - 4(\det \rho)}}{2}, \quad (21)$$

$$|\lambda_{\pm}\rangle = \pm v_{\pm} e^{i\tau} |A\rangle + v_{\mp} |A_{\perp}\rangle, \quad (22)$$

where $\det \rho = N_\alpha^4(1 - p_t^2)(1 - p_r^2)$ is the determinant of ρ , $|A\rangle$ and $|A_{\perp}\rangle$ are orthogonal basis vectors for the matrix of ρ , and

$$v_{\pm} = \left(\frac{1}{2} \pm \frac{\langle \sigma_z \rangle_{\rho}}{2\sqrt{1 - 4(\det \rho)}} \right)^{\frac{1}{2}}, \quad (23)$$

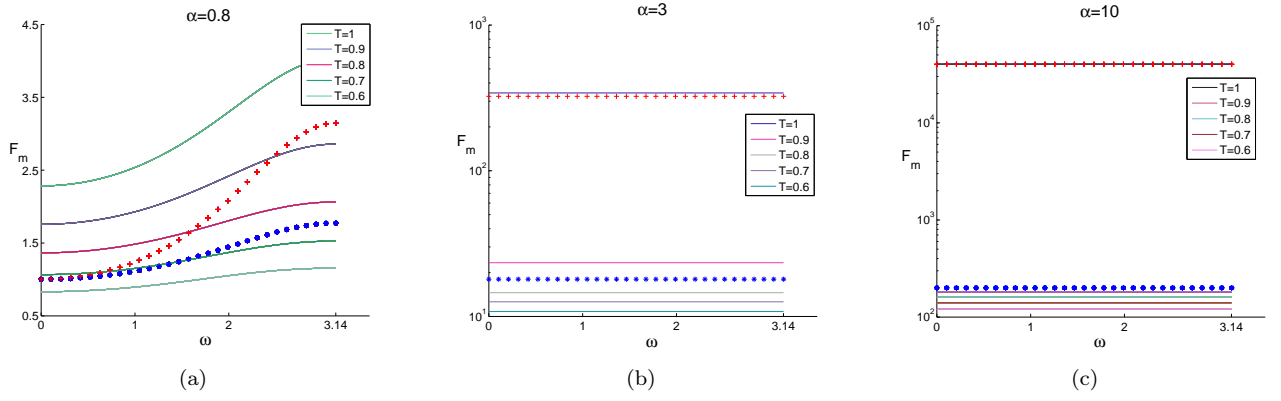


Figure 2: (Color online) The maximal QFI under the PMC. For each figure α is fixed, and is chosen to be 0.8, 3 and 10, separately. As α increases, the total photon number N also increases. The '+' dots represent N^2 while the '*' dots represent N as functions of ω . Each graph has five lines representing the F_m , which separately has, from bottom to top, the transmission coefficient T varying from 0.6 to 1. A line that is above the '+' dots indicates a surpass of the Heisenberg limit, while a line that is above the '*' dots indicates a surpass of the standard quantum limit (SQL).

with $\langle \sigma_z \rangle_\rho = 2\eta - 1 = 1 - 2N_\alpha^2(1 - p_t^2)$. Then through some calculation (see Appendix B) we have the analytic expression of the QFI as

$$\begin{aligned}
 F = & 2T|\alpha|^2(X + 1) + 4T^2|\alpha|^4X \\
 & + 4T^2|\alpha|^4 \cos^2 \phi \left(Z + 2\mu \langle \sigma_z \rangle_\rho \cos \tau - \mu^2 \frac{Z_1}{Z} \right) \\
 & - 4T^2|\alpha|^4 \sin^2 \phi \mu^2 \frac{Z_2}{Z} \\
 & - 4T^2|\alpha|^4 \sin(2\phi) (\langle \sigma_z \rangle_\rho - \mu \cos \tau) \mu \sin \tau, \quad (24)
 \end{aligned}$$

where $X = 2p_t \left(N_\alpha^2 p_t - \cos \tau \sqrt{N_\alpha^2 [1 - N_\alpha^2 (2 - p_t^2 - p_r^2)]} \right)$, $\mu = \sqrt{Z} p_t / \sqrt{1 - p_t^2}$, $Z = 1 - \langle \sigma_z \rangle_\rho^2 - 4(\det \rho)$, $Z_1 = (1 - \langle \sigma_z \rangle_\rho^2) \cos^2 \tau + 4(\det \rho) \sin^2 \tau$, $Z_2 = (1 - \langle \sigma_z \rangle_\rho^2) \sin^2 \tau + 4(\det \rho) \cos^2 \tau$.

To have a more intuitional vision of the result, we have plotted it as shown in Fig. 1. From this figure we can see that the analytic results coincide well with the numerical ones as the transmission coefficient T changes, the PMC does remain the same where $\phi = 0$. More importantly, such PMC invariance will be achieved both for input states with and without parity, that is, regardless of the value of ω . When $\phi = 0$, the maximal QFI F_m becomes

$$\begin{aligned}
 F_m = & 2T|\alpha|^2(X + 1) + 4T^2|\alpha|^4Z \\
 & + 4T^2|\alpha|^4 \left(X + 2\mu \langle \sigma_z \rangle_\rho \cos \tau - \mu^2 \frac{Z_1}{Z} \right). \quad (25)
 \end{aligned}$$

In Fig. 2 we have plotted F_m under different particle loss situation against ω as well as the corresponding N and N^2 . From the figure (a) we can see that with low intensity of input states, the QFI can lead to the surpass of the Heisenberg limit even photon losses are involved. As the total photon number N increases, in figure (b) the Heisenberg limit almost can only be beaten by the QFI

without any photon losses. However, it is still possible to surpass the SQL when T is smaller than 1. From figure (a) to figure (b) we can see that as $|\alpha|$ (or N) increases, F_m gradually becomes independent on ω , that is, when α is large enough, the parity of input states almost does not affect the value of the QFI. As N continue increasing (see figure (c)), the difference between with and without particle losses becomes more and more significant, even a small portion (10%) of photon leak will cause the QFI to decrease dramatically. The difference between figure (b) and (c) also indicates that the higher $|\alpha|$ (or N) is, the more difficult it is to surpass the standard quantum limit (SQL).

Now we consider what will happen to each term of F_m as $|\alpha|$ becomes larger and larger. This will give us explanations for the phenomena mentioned above. When $T \neq 1$, both p_r and p_t go to 0, and N_α^2 goes to 1/2. Consequently, all terms of F_m that have the factor $4T|\alpha|^4$ go to 0, so $F_m \approx 2T|\alpha|^2$, which is independent of ω , as we have concluded from Fig. 2. When $T=1$, however, $R=0$ and $p_r=1$, and the value of $\det \rho$ changes significantly. Recall that $\det \rho = N_\alpha^4 (1 - p_t^2) (1 - p_r^2)$, so as T changes from a number that is slightly smaller than 1 to exact "1", $\det \rho$ instantly changes from 1/4 to 0. This leads to a nonzero term with the factor $4T|\alpha|^4$, that is $Z(=1)$. Therefore, when $T=1$, $F_m \approx 2|\alpha|^2 + 4|\alpha|^4 \approx 4|\alpha|^4$, which is also independent of ω , as expected. It is also not difficult to see from Eq. (14) that with large $|\alpha|$, $N^2 \approx 4|\alpha|^4$. This means F_m and N^2 are almost the same with high intensity of input state, as shown in Fig. 2. Moreover, the abrupt change of the F_m caused by T also explains why that even a small portion of photon losses can lead to a significant decrease of the QFI.

Specifically, when $\omega = 0$, τ becomes zero, and QFI

becomes

$$F = 4T|\alpha|^2 [N_\alpha'^2 + T|\alpha|^2 (2N_\alpha'^2 - 1)] + 4T^2|\alpha|^4 \cos^2 \phi [1 - 4N_\alpha'^4 (1 - p_r^2)] - 16T^2|\alpha|^4 \sin^2 \phi N_\alpha'^4 (1 - p_r^2) p_t^2 \quad (26)$$

where $N_\alpha'^2 = 1/(2 + 2e^{-2|\alpha|^2})$. This is the same as the result in Ref. [8].

VI. CONCLUSION

In summary, we have considered a general scenario of a Mach-Zehnder interferometer. The input state in one mode is a coherent state and the other is a coherent superposition state whose parity can be controlled by an adjustable parameter. We have discussed both scenarios with and without particle losses and provided analytic expressions of the QFI under both circumstances. We have also shown the numeric results of the QFI with particle losses. The results indicate that the PMC is independent of phase difference between input states of the two modes and the parity of input states will not affect the QFI provided high intensity of input states is guaranteed.

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Appendix A: Express the Reduced Density Matrix ρ in Matrix Form

In this appendix we will utilize the following formula:

$$B^T|\alpha\rangle|\beta\rangle = |\alpha\sqrt{T} + i\beta\sqrt{R}\rangle|\beta\sqrt{T} + i\alpha\sqrt{R}\rangle, \quad (A.1)$$

where $B^T = \exp[i(2 \arccos \sqrt{T})J_x]$, $|\alpha\rangle$ and $|\beta\rangle$ are coherent states of mode A and mode B. This formula can be proven by expressing $|\alpha\rangle$ in the form

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} |0\rangle, \quad (A.2)$$

and the same goes with $|\beta\rangle$. Then taking advantage of the Hermiticity of B^T as well as the equation

$$B^\tau a^\dagger B^{\tau\dagger} = a^\dagger \cos \frac{\tau}{2} + ib^\dagger \sin \frac{\tau}{2}, \quad (A.3)$$

where $\tau = 2 \arccos \sqrt{T}$, one will find it not difficult to derive Eq. (A.1).

Equipping with this useful formula, we can now begin the calculation. Denote

$$|\text{in}\rangle = |i\alpha e^{i\phi}\rangle_A \otimes N_\alpha(|\alpha\rangle + e^{i\omega}|\alpha\rangle)_B, \quad (A.4)$$

then after the first beam splitter we have

$$\begin{aligned} & B^{\frac{1}{2}}|\text{in}\rangle \\ &= B^{\frac{1}{2}}|i\alpha e^{i\phi}\rangle_A \otimes N_\alpha(|\alpha\rangle + e^{i\omega}|\alpha\rangle)_B \\ &= N_\alpha \left| \frac{i\alpha}{\sqrt{2}}(1 + e^{i\phi}) \right\rangle_A \left| \frac{\alpha}{\sqrt{2}}(1 - e^{i\phi}) \right\rangle_B \\ &\quad + N_\alpha e^{i\omega} \left| \frac{-i\alpha}{\sqrt{2}}(1 - e^{i\phi}) \right\rangle_A \left| \frac{-\alpha}{\sqrt{2}}(1 + e^{i\phi}) \right\rangle_B. \end{aligned} \quad (A.5)$$

After that the state undergoes photon losses process, and we have

$$\begin{aligned} & B_{AC}^T B_{BD}^T B^{\frac{1}{2}}|\text{in}\rangle \otimes |00\rangle_{CD} \\ &= N_\alpha \left| \frac{i\alpha}{\sqrt{2}}(1 + e^{i\phi})\sqrt{T} \right\rangle_A \left| \frac{\alpha}{\sqrt{2}}(1 - e^{i\phi})\sqrt{T} \right\rangle_B \\ &\quad \otimes \left| \frac{-\alpha}{\sqrt{2}}(1 + e^{i\phi})\sqrt{R} \right\rangle_C \left| \frac{i\alpha}{\sqrt{2}}(1 - e^{i\phi})\sqrt{R} \right\rangle_D \\ &\quad + N_\alpha e^{i\omega} \left| \frac{-i\alpha}{\sqrt{2}}(1 - e^{i\phi})\sqrt{T} \right\rangle_A \left| \frac{-\alpha}{\sqrt{2}}(1 + e^{i\phi})\sqrt{T} \right\rangle_B \\ &\quad \otimes \left| \frac{\alpha}{\sqrt{2}}(1 - e^{i\phi})\sqrt{R} \right\rangle_C \left| \frac{-i\alpha}{\sqrt{2}}(1 + e^{i\phi})\sqrt{R} \right\rangle_D. \end{aligned} \quad (A.6)$$

To have a more concise and compact form we denote that

$$\begin{aligned} |A\rangle &= \left| \frac{i\alpha}{\sqrt{2}}(1 + e^{i\phi})\sqrt{T} \right\rangle_A \left| \frac{\alpha}{\sqrt{2}}(1 - e^{i\phi})\sqrt{T} \right\rangle_B, \\ |B\rangle &= \left| \frac{-i\alpha}{\sqrt{2}}(1 - e^{i\phi})\sqrt{T} \right\rangle_A \left| \frac{-\alpha}{\sqrt{2}}(1 + e^{i\phi})\sqrt{T} \right\rangle_B, \end{aligned} \quad (A.7)$$

then we will have

$$\begin{aligned} \rho &= \text{Tr}_{CD} \left(B_{AC}^T B_{BD}^T B^{\frac{1}{2}}|\text{in}\rangle \langle \text{in}| B^{\frac{1}{2}\dagger} B_{BD}^{T\dagger} B_{AC}^{T\dagger} \right) \\ &= N_\alpha^2 |A\rangle \langle A| + N_\alpha^2 p_r e^{-i\omega} |A\rangle \langle B| \\ &\quad + N_\alpha^2 p_r e^{i\omega} |B\rangle \langle A| + N_\alpha^2 |B\rangle \langle B|. \end{aligned} \quad (A.8)$$

Since $|A\rangle$ and $|B\rangle$ are not orthogonal, here we construct a new basis vector with the linear combination of $|A\rangle$ and $|B\rangle$ as

$$|A_\perp\rangle = \frac{1}{\sqrt{1 - p_t^2}} (|B\rangle - p_t |A\rangle), \quad (A.9)$$

which is orthogonal to $|A\rangle$. Then we have

$$\begin{aligned} \rho &= N_\alpha^2 (1 + 2e^{-2|\alpha|^2} \cos \omega + p_t^2) |A\rangle \langle A| \\ &\quad + N_\alpha^2 (p_r e^{-i\omega} + p_t) \sqrt{1 - p_t^2} |A\rangle \langle A_\perp| \\ &\quad + N_\alpha^2 (p_r e^{i\omega} + p_t) \sqrt{1 - p_t^2} |A_\perp\rangle \langle A| \\ &\quad + N_\alpha^2 (1 - p_t^2) |A_\perp\rangle \langle A_\perp|. \end{aligned} \quad (A.10)$$

Now we can obtain the matrix form of ρ as shown in Eq. (19).

Appendix B: Calculation of the QFI Under Particle Losses

In this appendix we show the calculation details of the QFI with Photon Losses. Firstly, we write down Eq. (20) without abbreviation as

$$F = 4\lambda_+ \langle \lambda_+ | J_z^2 | \lambda_+ \rangle + 4\lambda_- \langle \lambda_- | J_z^2 | \lambda_- \rangle - 4\lambda_+ |\langle \lambda_+ | J_z | \lambda_+ \rangle|^2 - 4\lambda_- |\langle \lambda_- | J_z | \lambda_- \rangle|^2 - 16\lambda_+ \lambda_- |\langle \lambda_+ | J_z | \lambda_- \rangle|^2 \quad (\text{B.1})$$

Then according to Eq. (22) and Eq. (A.9), we have

$$|\lambda_+\rangle = \left(v_+ e^{i\tau} - \frac{p_t v_-}{\sqrt{1-p_t^2}} \right) |A\rangle + \frac{v_-}{\sqrt{1-p_t^2}} |B\rangle, \quad (\text{B.2})$$

$$|\lambda_-\rangle = \left(-v_- e^{i\tau} - \frac{p_t v_+}{\sqrt{1-p_t^2}} \right) |A\rangle + \frac{v_+}{\sqrt{1-p_t^2}} |B\rangle. \quad (\text{B.3})$$

Notice that we first transfer from $|A\rangle - |B\rangle$ basis to $|A\rangle - |A_\perp\rangle$ basis so as to obtain the eigenvalues and eigenvectors of the reduced density matrix ρ . And now we go back to the $|A\rangle - |B\rangle$ basis, because the calculation is more straightforward in this basis.

Then we calculate the terms of F separately. First we have

$$\begin{aligned} & \langle \lambda_+ | J_z^2 | \lambda_+ \rangle \\ &= \left(v_+ e^{i\tau} - \frac{p_t v_-}{\sqrt{1-p_t^2}} \right) \left(v_+ e^{-i\tau} - \frac{p_t v_-}{\sqrt{1-p_t^2}} \right) \langle A | J_z^2 | A \rangle \\ &+ \frac{v_-^2}{1-p_t^2} \langle B | J_z^2 | B \rangle \\ &+ \left(v_+ e^{-i\tau} - \frac{p_t v_-}{\sqrt{1-p_t^2}} \right) \frac{v_-}{\sqrt{1-p_t^2}} \langle A | J_z^2 | B \rangle \\ &+ \left(v_+ e^{i\tau} - \frac{p_t v_-}{\sqrt{1-p_t^2}} \right) \frac{v_-}{\sqrt{1-p_t^2}} \langle B | J_z^2 | A \rangle \end{aligned} \quad (\text{B.4})$$

Since $|A\rangle$ and $|B\rangle$ are direct products of coherent states in mode A and mode B, it is not difficult to find that

$$\langle A | J_z^2 | A \rangle = \frac{1}{2} T |\alpha|^2 + T^2 |\alpha|^4 \cos^2 \phi \quad (\text{B.5})$$

and

$$\begin{aligned} \langle B | J_z^2 | B \rangle &= \frac{1}{2} T |\alpha|^2 + T^2 |\alpha|^4 \cos^2 \phi \\ &= \langle A | J_z^2 | A \rangle. \end{aligned} \quad (\text{B.6})$$

Also we have

$$\langle A | J_z^2 | B \rangle = \langle B | J_z^2 | A \rangle = -p_t T^2 |\alpha|^4 \sin^2 \phi. \quad (\text{B.7})$$

Before substituting the expressions of $\langle A | J_z^2 | A \rangle$ and other three terms into Eq. (B.4), we first simplify it as

$$\begin{aligned} & \langle \lambda_+ | J_z^2 | \lambda_+ \rangle \\ &= \left(v_+^2 + \frac{(1+p_t^2)v_-^2}{1-p_t^2} - 2 \frac{p_t v_+ v_-}{\sqrt{1-p_t^2}} \cos \tau \right) \langle A | J_z^2 | A \rangle \\ &+ \left(2 \frac{v_+ v_-}{\sqrt{1-p_t^2}} \cos \tau - 2 \frac{p_t v_-^2}{1-p_t^2} \right) \langle A | J_z^2 | B \rangle. \end{aligned} \quad (\text{B.8})$$

In like manner, we can have

$$\begin{aligned} & \langle \lambda_- | J_z^2 | \lambda_- \rangle \\ &= \left(v_-^2 + \frac{(1+p_t^2)v_+^2}{1-p_t^2} + 2 \frac{p_t v_+ v_-}{\sqrt{1-p_t^2}} \cos \tau \right) \langle A | J_z^2 | A \rangle \\ &- \left(2 \frac{v_+ v_-}{\sqrt{1-p_t^2}} \cos \tau + 2 \frac{p_t v_+^2}{1-p_t^2} \right) \langle A | J_z^2 | B \rangle. \end{aligned} \quad (\text{B.9})$$

Then the first part of the QFI becomes

$$\begin{aligned} & 4\lambda_+ \langle \lambda_+ | J_z^2 | \lambda_+ \rangle + 4\lambda_- \langle \lambda_- | J_z^2 | \lambda_- \rangle \\ &= 4 \left[(\lambda_+ v_+^2 + \lambda_- v_-^2) + (\lambda_+ v_-^2 + \lambda_- v_+^2) \frac{1+p_t^2}{1-p_t^2} \right. \\ &\quad \left. - 2(\lambda_+ - \lambda_-) \frac{p_t v_+ v_-}{\sqrt{1-p_t^2}} \cos \tau \right] \langle A | J_z^2 | A \rangle \\ &+ 8 \left[(\lambda_+ - \lambda_-) \frac{v_+ v_-}{\sqrt{1-p_t^2}} \cos \tau \right. \\ &\quad \left. - (\lambda_+ v_-^2 + \lambda_- v_+^2) \frac{p_t}{\sqrt{1-p_t^2}} \right] \langle A | J_z^2 | B \rangle \\ &= 2T |\alpha|^2 (1 + 2N_\alpha^2 p_t^2 - 2p_t \mathcal{M} \cos \tau) \\ &\quad + 4T^2 |\alpha|^4 (2N_\alpha^2 p_t^2 - 2p_t \mathcal{M} \cos \tau) \\ &\quad + 4T^2 |\alpha|^4 \cos^2 \phi. \end{aligned} \quad (\text{B.10})$$

In the second step of Eq. (B.10), we have utilized the following conclusions and notations:

$$\lambda_\pm = \frac{1 \pm \sqrt{1 - 4\det\rho}}{2}, \quad (\text{B.11})$$

$$v_\pm = \left(\frac{1}{2} \pm \frac{\langle \sigma_z \rangle_\rho}{2\sqrt{1 - 4\det\rho}} \right)^{\frac{1}{2}}, \quad (\text{B.12})$$

$$\lambda_+ - \lambda_- = \sqrt{1 - 4\det\rho}, \quad (\text{B.13})$$

$$\lambda_+ v_+^2 + \lambda_- v_-^2 = 1 - N_\alpha^2 (1 - p_t^2), \quad (\text{B.14})$$

$$\lambda_+ v_-^2 + \lambda_- v_+^2 = N_\alpha^2 (1 - p_t^2), \quad (\text{B.15})$$

$$\begin{aligned} \mathcal{M} &= \frac{(\lambda_+ - \lambda_-) v_+ v_-}{\sqrt{1 - p_t^2}} \\ &= \sqrt{N_\alpha^2 [1 - N_\alpha^2 (2 - p_t^2 - p_r^2)]}, \end{aligned} \quad (\text{B.16})$$

Now we continue to calculate the second part of the

QFI. Firstly,

$$\begin{aligned}
& \langle \lambda_+ | J_z | \lambda_+ \rangle \\
&= \left(v_+ e^{i\tau} - \frac{p_t v_-}{\sqrt{1-p_t^2}} \right) \left(v_+ e^{-i\tau} - \frac{p_t v_-}{\sqrt{1-p_t^2}} \right) \langle A | J_z | A \rangle \\
&+ \frac{v_-^2}{1-p_t^2} \langle B | J_z | B \rangle \\
&+ \left(v_+ e^{-i\tau} - \frac{p_t v_-}{\sqrt{1-p_t^2}} \right) \frac{v_-}{\sqrt{1-p_t^2}} \langle A | J_z | B \rangle \\
&+ \left(v_+ e^{i\tau} - \frac{p_t v_-}{\sqrt{1-p_t^2}} \right) \frac{v_-}{\sqrt{1-p_t^2}} \langle B | J_z | A \rangle, \quad (B.17)
\end{aligned}$$

and we can also find

$$\begin{aligned}
\langle A | J_z | A \rangle &= T|\alpha|^2 \cos \phi - T|\alpha|^2 \cos \phi \\
&= -\langle B | J_z | B \rangle, \quad (B.18)
\end{aligned}$$

$$\begin{aligned}
\langle A | J_z | B \rangle &= ip_t T|\alpha|^2 \sin \phi \\
&= -\langle B | J_z | A \rangle. \quad (B.19)
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \langle \lambda_+ | J_z | \lambda_+ \rangle \\
&= T|\alpha|^2 \cos \phi \left(v_+^2 - v_-^2 - 2 \frac{p_t v_+ v_-}{\sqrt{1-p_t^2}} \cos \tau \right) \\
&+ 2T|\alpha|^2 \sin \phi \frac{p_t v_+ v_-}{\sqrt{1-p_t^2}} \sin \tau, \quad (B.20)
\end{aligned}$$

and

$$\begin{aligned}
& \langle \lambda_- | J_z | \lambda_- \rangle \\
&= T|\alpha|^2 \cos \phi \left(-v_+^2 + v_-^2 + 2 \frac{p_t v_+ v_-}{\sqrt{1-p_t^2}} \cos \tau \right) \\
&- 2T|\alpha|^2 \sin \phi \frac{p_t v_+ v_-}{\sqrt{1-p_t^2}} \sin \tau \\
&= -\langle \lambda_+ | J_z | \lambda_+ \rangle, \quad (B.21)
\end{aligned}$$

which indicates $|\langle \lambda_+ | J_z | \lambda_+ \rangle|^2 = |\langle \lambda_- | J_z | \lambda_- \rangle|^2$. Now we

can obtain the second part of the QFI as

$$\begin{aligned}
& -4\lambda_+ |\langle \lambda_+ | J_z | \lambda_+ \rangle|^2 - 4\lambda_- |\langle \lambda_- | J_z | \lambda_- \rangle|^2 \\
&= -4(\lambda_+ + \lambda_-) |\langle \lambda_+ | J_z | \lambda_+ \rangle|^2 \\
&= -4T^2 |\alpha|^4 \cos^2 \phi \left[(v_+^2 - v_-^2)^2 + 4 \frac{p_t^2 v_+^2 v_-^2}{1-p_t^2} \cos^2 \tau \right. \\
&\quad \left. - 4(v_+^2 - v_-^2) \frac{p_t v_+ v_-}{\sqrt{1-p_t^2}} \cos \tau \right] \\
&- 4T^2 |\alpha|^4 \sin^2 \phi \left(4 \frac{p_t^2 v_+^2 v_-^2}{1-p_t^2} \sin^2 \tau \right) \\
&+ 4T^2 |\alpha|^4 \sin(2\phi) \left[2 \frac{p_t^2 v_+^2 v_-^2}{1-p_t^2} \sin 2\tau \right. \\
&\quad \left. - 2(v_+^2 - v_-^2) \frac{p_t v_+ v_-}{\sqrt{1-p_t^2}} \sin \tau \right]. \quad (B.22)
\end{aligned}$$

Then we come to the last part of the QFI. Firstly

$$\begin{aligned}
& \langle \lambda_+ | J_z | \lambda_- \rangle \\
&= \left(v_+ e^{-i\tau} - \frac{p_t v_-}{\sqrt{1-p_t^2}} \right) \left(-v_- e^{i\tau} - \frac{p_t v_+}{\sqrt{1-p_t^2}} \right) \langle A | J_z | A \rangle \\
&+ \frac{v_+ v_-}{1-p_t^2} \langle B | J_z | B \rangle \\
&+ \left(v_+ e^{-i\tau} - \frac{p_t v_-}{\sqrt{1-p_t^2}} \right) \frac{v_+}{\sqrt{1-p_t^2}} \langle A | J_z | B \rangle \\
&+ \left(-v_- e^{i\tau} - \frac{p_t v_+}{\sqrt{1-p_t^2}} \right) \frac{v_-}{\sqrt{1-p_t^2}} \langle B | J_z | A \rangle \\
&= T|\alpha|^2 \cos \phi \left(-2v_+ v_- - \frac{p_t v_+^2}{\sqrt{1-p_t^2}} e^{-i\tau} + \frac{p_t v_-^2}{\sqrt{1-p_t^2}} e^{i\tau} \right) \\
&+ ip_t T|\alpha|^2 \sin \phi \left(\frac{v_+^2}{\sqrt{1-p_t^2}} e^{-i\tau} + \frac{v_-^2}{\sqrt{1-p_t^2}} e^{i\tau} \right). \quad (B.23)
\end{aligned}$$

In the second step we have utilized Eq. (B.18) and

Eq. (B.19). Then

$$\begin{aligned}
& -16\lambda_+\lambda_-|\langle\lambda_+|J_z|\lambda_- \rangle|^2 \\
& = -16(\det\rho)T^2|\alpha|^4 \\
& \times \left\{ \cos^2\phi \left[4v_+^2v_-^2 + \frac{p_t^2}{1-p_t^2}\sin^2\tau \right. \right. \\
& \quad + 4(v_+^3v_- - v_-^3v_+)\frac{p_t}{\sqrt{1-p_t^2}}\cos\tau \\
& \quad \left. + (v_+^2 - v_-^2)^2\frac{p_t^2}{1-p_t^2}\cos^2\tau \right] \\
& + \sin^2\phi \left[(v_+^2 - v_-^2)^2\frac{p_t^2}{1-p_t^2}\sin^2\tau + \frac{p_t^2}{1-p_t^2}\cos^2\tau \right] \\
& + \sin(2\phi) \left[(1 - (v_+^2 - v_-^2)^2)\frac{p_t^2}{1-p_t^2}\sin\tau\cos\tau \right. \\
& \quad \left. - 2(v_+^3v_- - v_-^3v_+)\frac{p_t}{\sqrt{1-p_t^2}}\sin\tau \right] \Big\}. \tag{B.24}
\end{aligned}$$

With all of these three parts, we can now obtain the final expression of the QFI under photon losses, that is Eq. (24).

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